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Additivity of Vector Gleason Measures

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We study the degree of additivity of orthogonal Hilbert-space-valued measures on the lattice $L(H)$ of all projections acting on a Hilbert space H. We present criteria for such measures to be completely additive and we establish the connection between the additivity of orthogonal measures and the size of almost disjoint families on dim H. [For example, we show that every H-valued orthogonal measure is weakly σ-additive iff $(\dim H)^{\omega} > \dim H$.] As a corollary we see that finitely additive orthogonal measures distinguish dimensions of Hilbert spaces (this can be viewed as a generalization of a theorem by Kruszynski). As a further corollary, we obtain that, for cardinals κ, ν with $\kappa > \nu$, $\kappa \geq 3$, there is no Jordan homomorphism from a type I_{κ}-factor into a type I_{λ}-factor. Finally, we show that every lattice $L(H)$ with $(\dim H)^{\omega} = \dim H$ admits a nonzero free orthogonal measure with values in H. Our results contribute to the noncommutative probability theory and also may find applications in the theory of the representation of C^*-algebras.

1. INTRODUCTION AND PRELIMINARIES

The objective of this paper is to strengthen hitherto known results about the additivity of orthogonal vector measures obtained in Hamhalter (to appear) and Kruszynski (1988). Measures of this type are closely related to the operator theory, Hilbert space geometry, noncommutative probability theory, and generalized function theory (Goldstein, 1991; Hamhalter, 1991; Jajte, 1979; Jajte and Paszkiewicz, 1978; Kruszynski, 1988, 1990; Masani, 1970). They also play a significant role in the foundations of quantum theory (Kruszynski, 1990; Pták and Pulmannová, 1991; Varadarajan, 1968).

We shall be mainly interested in the question of when orthogonal measures are completely additive or σ-additive. Problems of this kind often occur in classical as well as noncommutative measure theory (Alexandroff, 1941; Béaver and Cook, 1977; Dvurečenskij *et al.*, 1991; De Lucia and Morales,

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to appear). It has been proved in Hamhalter (to appear) that every finitely additive orthogonal measure $m\colon L(H) \to H$ is completely additive whenever H is separable. This result cannot be generalized for every nonseparable space (Proposition 2.9). We show that its validity is relevant to combinatorial properties of $\dim H$. Loosely speaking, it turns out that the size of the range of an orthogonal measure determines its degree of additivity. It has been shown (Kruszynski, 1988) that $\dim K \geq \dim H$ if and only if $L(H)$ admits a nonzero completely additive orthogonal measure with values in K. Our considerations yield a somewhat surprising strengthening of this result: We have $\dim K \geq \dim H$ if and only if there is a nonzero orthogonal measure $m\colon L(H) \to K$. As a consequence, we obtain that there is no Jordan homomorphism $\phi\colon \mathscr{B}(H) \to \mathscr{B}(K)$ ($\dim H \geq 3$), where $\mathscr{B}(H)$, $\mathscr{B}(K)$ are the C^*-algebras of all bounded operators on H, K, and $\dim K < \dim H$. [Thus, for instance, $\mathscr{B}(H)$ cannot be represented on a Hilbert space with strictly smaller dimension.] This result has been hitherto known only for finite-dimensional spaces (Kruszynski, 1988). Results about the nonexistence of nonzero orthogonal measures may be also relevant to the hidden parameter problem in quantum mechanics (Kruszynski, 1990; Manin, 1977). In the conclusion of this paper we construct a nontrivial "free" orthogonal measure $m\colon L(H) \to H$ provided that $(\dim H)^{\omega} = \dim H$.

The paper is divided into two parts. Section 1.1 is devoted to a short analysis of almost disjoint families of sets which are an important technical tool for further considerations. In Section 1.2 some auxiliary results about nonnegative Gleason measures are proved. The main new results are then presented in Section 2.

1.1. Almost Disjoint Families of Sets

Let κ be a cardinal number. In this section we review some results of the set theory we shall use in the sequel. Our general reference here is Balcar and Štěpánek (1986). Let κ be a cardinal number. By κ^+ and $\mathrm{cf}(\kappa)$ we shall denote a successor and a cofinal of κ, respectively. Let $\mathscr{P}(\kappa)$ denote the power set of κ and let $[\kappa]^{\nu}$ be the set consisting of all elements of $\mathscr{P}(\kappa)$ with the cardinality equal to ν and let ${}^{\nu}\kappa$ mean the set of all mappings of ν into κ. We use the notation κ^{ν} for $|{}^{\nu}\kappa|$. It is well known that $|[\kappa]^{\nu}| = \kappa^{\nu}$ for any $\nu \leq \kappa$.

Definition 1.1. A family $S \subset [\kappa]^{\nu}$ (resp. $S \subset [\kappa]^{\kappa}$) is called ν-almost disjoint (resp. strongly ν-almost disjoint) if $|x \cap y| < \nu$, whenever $x, y \in S$ and $x \neq y$.

Proposition 1.2. Suppose that $\kappa \geq \omega$. Then κ admits an ω-almost disjoint family S such that $|S| > \kappa$ if and only if $\kappa^{\omega} > \kappa$.

Proof. If S is an ω-almost disjoint family and $\kappa^{\omega}=\kappa$, then, obviously, $|S|\leq|[\kappa]^{\omega}|=\kappa^{\omega}=\kappa$.

Suppose that $\kappa^{\omega}>\kappa$. Let us define for every $f\in{}^{\omega}\kappa$ a mapping $\hat{f}\in{}^{\omega}(\bigcup_{n\in\omega}{}^{n}\kappa)$ by putting $\hat{f}=f|n$ $(n\in\omega)$. If $f\neq g$, then there is an $n_0\in\omega$ such that $f(n_0)\neq g(n_0)$. Then $\hat{f}(n)\neq\hat{g}(n)$ for every $n\geq n_0$. Put $S_f=Rg\hat{f}$. (Here $Rg\hat{f}$ denotes the range of $\hat{f}$.) It follows that $|S_f|=\omega$ and $S_f\cap S_g|<\omega$ for any $f\neq g$. Because $|\bigcup_{n\in\omega}{}^{n}\kappa|=\kappa$, we can identify κ with $\bigcup_{n\in\omega}{}^{n}\kappa$ and view $S=(S_f)_{f\in{}^{\omega}\kappa}$ as an ω-almost disjoint family on κ. We see that $|S|=\kappa^{\omega}>\kappa$. ■

Let us remark that, if $\mathrm{cf}(\kappa)=\omega$, then $\kappa^{\omega}>\kappa$. The reverse implication holds if we admit the generalized continuum hypothesis (Balcar and Štěpánek, 1986).

1.2. Additivity of Orthogonal Vector Measures

Let us first fix some notations. Throughout the paper let H be an arbitrary (complex) Hilbert space with $\dim H\geq 3$. Let $\mathscr{B}(H)$ stand for the C^*-algebra of all bounded operators acting on H (with identity I). For $A\in\mathscr{B}(H)$ let us denote by $R(A)$ the range projection of A. Let $S(H)$ and $L(H)$ mean the sets of self-adjoint operators and projections acting on H, respectively. If $x\in H$, then P_x denotes a projection with $P_x(H)=\mathrm{sp}\{x\}$. As is known (see, e.g., Pták and Pulmannová, 1991; Varadarajan, 1968), $L(H)$ forms an orthomodular lattice with an orthocomplementation $P^{\perp}=I-P$.

A mapping $s\colon L(H)\to[0,\infty)$ is said to be a *Gleason measure* (abbr. a *measure*) if $\sup_{P\in L(H)}|s(P)|<\infty$ and if $s(P+Q)=s(P)+s(Q)$, whenever $P\perp Q$ $[P, Q\in L(H)]$. Let us introduce the following classes of measures.

Definition 1.3. A measure s on $L(H)$ is said to be κ-additive (resp. weakly κ-additive) $[\kappa\leq(\dim H)^{+}]$ if

$$s\left(\sum_{\alpha\in I}P_\alpha\right)=\sum_{\alpha\in I}s(P_\alpha)$$

whenever $(P_\alpha)_{\alpha\in I}$ is a family of mutually orthogonal projections (resp. finite-dimensional projections) with $|I|<\kappa$.

Let s be a measure on $L(H)$. By our definition, s is ω-additive (or, equivalently, s is finitely additive). We say that s is *σ-additive* (resp. *completely additive*) if it is ω_1-additive [resp. if it is $(\dim H)^{+}$-additive]. The same convention is used for weakly additive measures. Clearly, s is completely additive if and only if it is $(\dim H)^+$-weakly additive. A measure s is said to be *free* if it vanishes on all finite-dimensional projections. Further, s is called *weakly regular* if, for any $P \in L(H)$ with $s(P) > 0$ there is a finite-dimensional projection $Q \leq P$ such that $s(Q) > 0$ (Dvurečenskij, 1990; De Lucia and Morales, to appear). Obviously, every completely additive measure is weakly regular and the reverse implication does not have to be true. For instance, let H be a separable Hilbert space with an orthonormal basis $(e_i)_{i \in N}$. Put $s_1 = \sum_{i \in N} 2^{-i} t_i$, where $t_i(P) = \|Pe_i\|^2$ $[P \in L(H)]$. If s_2 is any noncompletely additive measure on $L(H)$, then $s = s_1 + s_2$ is a weakly regular, noncompletely additive measure.

We say that a measure s has a *support* if there is $P \in L(H)$ such that the following condition is satisfied: $s(Q) = 0$ iff $Q \perp P$. It is well known that every σ-additive measure with a support has to be completely additive (Maeda, 1980). In Dvurečenskij (1990) an open problem has been posed of when a measure s on $L(H)$ which possesses a support has to be completely additive. The following assertion says that even a weakly regular measure s (which has a separable support) need not be completely additive. In this case we give an additional criterion for such a measure s to be completely additive.

Proposition 1.4. (i) Every weakly regular measure on $L(H)$ has a support.

(ii) Every weakly regular, weakly σ-additive measure on $L(H)$ is completely additive.

Proof. (i) Let s be a weakly regular measure on $L(H)$. According to Aarnes (1970), s can be decomposed into a sum in such a way that $s = s_1 + s_2$, where s_1 is a completely additive measure and s_2 is a measure vanishing on all finite-dimensional projections. Making use of Gleason's (1957) theorem, we can represent s_1 by a nonnegative trace-class operator $T \in \mathscr{B}(H)$ via the formula $s_1(P) = \mathrm{Tr}\, TP$ $[P \in L(H)]$. Put $M = \overline{\mathrm{sp}}(x_i)_{i \in N}$, where $(x_i)_{i \in N}$ is a sequence consisting of all nonzero eigenvectors of T. Then the projection P with the range M is a separable support for s_1. We shall show that P is also a support of s. For this, let us note first that $s(P^\perp) = 0$. Indeed, if $s(P^\perp) = s_2(P^\perp) \neq 0$, then there is a nonzero $x \in H$ such that $P_x \leq P^\perp$ and $s(P_x) = s_2(P_x) \neq 0$, which is a contradiction. On the other hand, if $s(Q) = 0$, then $s_1(Q) = 0$ and so $Q \perp P$.

(ii) Let s be a weakly regular, weakly σ-additive measure on $L(H)$. Using the notation of the preceding paragraph, we obtain that the support P of s coincides with the support of s_1. Also, $s_2(P) = 0$, because P is separable. Thus, $s_2(I) = s_2(P) + s_2(P^\perp) = 0$ and therefore s $(=s_1)$ is completely additive. ■

2. ORTHOGONAL VECTOR MEASURES

Throughout this section let K be an arbitrary Hilbert space.

Definition 2.1. A mapping $m: L(H) \to K$ is said to be an orthogonal Gleason measure (abbr. an orthogonal measure) if $m(P+Q) = m(P) + m(Q)$ and $m(P) \perp m(Q)$ for any orthogonal pair $P, Q \in L(H)$.

Let $m: L(H) \to K$ be an orthogonal measure. A mapping $s_m: L(H) \to [0, \infty)$ defined by putting $s_m(P) = \|m(P)\|^2$ $[P \in L(H)]$ is a measure on $L(H)$. Indeed, s_m is obviously additive and $s_m(P) \leq \|m(P)\|^2 + \|m(P^\perp)\|^2 = \|m(I)\|^2$ $[P \in L(H)]$. We say that m is *κ-additive* (resp. weakly *κ-additive*) if the corresponding measure s_m is κ-additive (resp. weakly κ-additive). In the same way we define a *weakly regular* and *free* orthogonal measure. By $\mathscr{R}(m)$ we shall denote the space $\overline{\text{sp}}\{m(P) | P \in L(H)\}$. Two orthogonal measures $m_1: L(H) \to K_1$ and $m_2: L(H) \to K_2$ are called *equivalent* if there is a unitary mapping $\mathscr{U}$ of $\mathscr{R}(m_1)$ onto $\mathscr{R}(m_2)$ such that $m_2 = \mathscr{U} \circ m_1$. Following Goldstein (1991), a bounded linear mapping $F: \mathscr{B}(H) \to K$ is said to be an *orthogonal vector field* on $\mathscr{B}(H)$ if $\langle F(P), F(Q) \rangle = 0$, whenever $P \perp Q$ $[P, Q \in L(H)]$. If F is an orthogonal vector field, then $F | L(H)$ is an orthogonal measure. As the following theorem says, one can conversely prove that every orthogonal measure arises this way [see Hamhalter (to appear) for the separable case].

Theorem 2.2. Let $m: L(H) \to K$ be an orthogonal measure. Then there is an orthogonal vector field $F: \mathscr{B}(H) \to K$ such that $F | L(H) = m$.

Proof (A sketch). Since the proof is essentially the same as in the separable case [given in Hamhalter (to appear)], we only outline the basic ideas. Fix an $A \in S(H)$ and denote by $L(A)$ the Boolean algebra consisting of all projections of the smallest Abelian von Neumann subalgebra of $\mathscr{B}(H)$ containing A. Then m extends to a linear mapping $F_A: \text{sp}\, L(A) \to K$. Simple computations show that

$$\|F_A(S)\| \leq \|S\| \, \|m(I)\| \qquad \text{for every } S \in \text{sp}\, L(A)$$

Thus, F_A admits a continuous linear extension over $\overline{\text{sp}}\, L(A)$. Making use of the spectral theorem, we can define $F(A) = F_A(A)$ and then $F(B) = F(\text{Re}\, B) + F(\text{Im}\, B)$ for general $B \in \mathscr{B}(H)$. By the preceding estimation we infer that

$$\|F(B)\| \leq 2 \|B\| \, \|m(I)\|$$

Having noted that F is bounded on the unit ball, it suffices to prove that F is linear. For this purpose let us define, for any $x \in H$, a mapping

$m_x: L(H) \to C$ such that

$$m_x(P) = \langle m(P), x\rangle, \qquad P \in L(H)$$

Then m_x is a bounded complex-valued measure on $L(H)$. According to a result of Matveichuk (1990), there is a bounded functional f_x of $\mathscr{B}(H)$ extending m_x. Moreover, if $A \in S(H)$, then $f_x(A) = \langle F(A), x\rangle$ (we again use the spectral theorem). Thus, $\langle F(A_1 + A_2), x\rangle = \langle F(A_1) + F(A_2), x\rangle$ for any $A_1, A_2 \in S(H)$ and $x \in K$. Hencefore F is a linear mapping. ■

Let us remark that Theorem 2.2 remains valid for any projection lattice of a von Neumann algebra without type I_2 direct summand.

It has been shown in Hamhalter (to appear) that the dimension of every range space $\mathscr{R}(m)$ of an orthogonal measure m is given by the corresponding measure s_m. For the convenience of the reader we outline the proof here.

Lemma 2.3. Let $m_1: L(H) \to K_1$ and $m_2: L(H) \to K_2$ be two orthogonal measures. If $s_{m_1} = s_{m_2}$, then $\dim \mathscr{R}(m_1) = \dim R(m_2)$.

Proof (A sketch). We extend m_1 and m_2 to a continuous linear mapping F_1 and F_2 on $\mathscr{B}(H)$. Employing the spectral theorem and the continuity of F_1 and F_2, it can be proved that $\|F_1(A)\| = \|F_2(A)\|$ for any $A \in S(H)$.

Let us define a mapping $V: F_1(S(H)) \to F_2(S(H))$ by putting $V(F_1(A)) = F_2(A)$ $[A \in S(H)]$. Then V is a well-defined isometry and therefore the sets $F_1(S(H))$ and $F_2(S(H))$ can be viewed as isometric metric spaces. Finally,

$$\mathscr{R}(m_j) = \overline{F_j(S(H)) + iF_j(S(H))} \qquad (j = 1, 2)$$

and so $\dim \mathscr{R}(m_1) = \dim \mathscr{R}(m_2)$. ■

The main result of this paper is based on the following lemma.

Lemma 2.4. Let $m: L(H) \to K$ be an orthogonal vector measure and let $S \subset \mathscr{P}(\dim H)$. Then $\dim K \geq |S|$ if either of the following conditions is satisfied:

(i) S is a strongly ω-almost disjoint family and m is not completely additive.

(ii) S is a ν-almost disjoint family and m is a weakly ν-additive measure which is not weakly ν^+-additive.

Proof. Let s_m be the corresponding real-valued measure defined above and let f be a positive functional of $\mathscr{B}(H)$ which extends s_m (Matveichuk, 1990). Using the GNS construction, we can find a Hilbert space $\mathscr{H}$ and a representation π such that

$$f(A)=\langle \pi(A)x, x\rangle \qquad \text{for any} \quad A\in \mathscr{B}(H)$$

where $x\in\mathscr{H}$ is a cyclic vector for π. Then $\|m(P)\|^2=f(P)=\|\pi(P)x\|^2$. According to Lemma 2.3, we may assume that $m(P)=\pi(P)x$ $[P\in L(H)]$. A representation π can be decomposed (up to unitary equivalence) into the direct sum $\pi_1\oplus\pi_2$, where $\pi_1:\mathscr{B}(H)\to\mathscr{B}(\sum_{\alpha\in I}\oplus H)=\mathscr{B}(\mathscr{H}_{\pi_1})$ is either constantly zero or it is a representation of the form

$$\pi_1(A)=\sum_{\alpha\in I}\oplus A, \qquad A\in\mathscr{B}(H)$$

and the mapping $\pi_2:\mathscr{B}(H)\to\mathscr{B}(\mathscr{H}_{\pi_2})$ is either constantly zero or it is a representation such that $\pi_2|\mathscr{K}=0$, where $\mathscr{K}$ is the ideal of all compact operators acting on H (Kadison and Ringrose, 1986). Let $x=x_1\oplus x_2$, where $x_1\in\mathscr{H}_{\pi_1}$ and $x_2\in\mathscr{H}_{\pi_2}$. Let $S\subset\mathscr{P}(\dim H)$ and let $(e_\beta)_{\beta\in J}$ be an orthonormal basis of H. Let us define a collection of mutually commuting projections $(P_\gamma)_{\gamma\in S}$ such that each P_γ is an orthogonal projection of H onto the space $\overline{\text{sp}}\,(e_\beta)_{\beta\in\gamma}$.

Suppose first that S is a strongly ω-almost disjoint family and m is not completely additive. Then the component π_2 in the decomposition of π has to be nonzero. Indeed, in the opposite case, $f(A)=\langle(\sum_{\alpha\in I}\oplus A)y, y\rangle$ for some $y\in\mathscr{H}_{\pi_2}$, which is a normal (i.e., completely additive) functional—a contradiction. If $\gamma_1\neq\gamma_2$, then $P_{\gamma_1}P_{\gamma_2}\in\mathscr{K}$ and so

$$\pi_2(P_{\gamma_1})\pi_2(P_{\gamma_2})=\pi_2(P_{\gamma_1}P_{\gamma_2})=0$$

On the other hand, we have $\pi_2(P_\gamma)\neq 0$ for every $\gamma\in S$. To see this, let us note that each P_γ projects onto a subspace with the dimension equal to $\dim H$. Thus, P_γ and I are equivalent projections [in $\mathscr{B}(H)$] and so $\pi_2(P_\gamma)\neq 0$, because the null space of π_2 is a two-sided ideal in $\mathscr{B}(H)$.

Suppose now that S is ν-almost disjoint family and that m is weakly ν-additive measure, which is not weakly ν^+-additive. As in the preceding paragraph, the mapping π_2 cannot be zero on every ν-dimensional projection, for otherwise $m(P)=\|\pi_1(P)y\|^2$ $(y\in\mathscr{H}_{\pi_1})$ whenever $\dim P(H)<\nu$—a contradiction with the assumption. Because all ν-dimensional projections are equivalent [in $\mathscr{B}(H)$], we see again that $\pi_2(P_\gamma)\neq 0$ for every $\gamma\in S$. On the other hand, let $P\in L(H)$ and $\dim P(H)<\nu$. We prove that $\pi_2(P)=0$. Suppose, on the contrary, that $\pi_2(P)\neq 0$. Then x_2 is a cyclic vector for the representation π_2. Let us define a functional f_2 by the equality $f_2(A)=$ $<\pi_2(A)x_2, x_2>$ $[A\in\mathscr{B}(H)]$. Put

$$\mathscr{H}_P=\pi(P)(\mathscr{H}_{\pi_2})=\overline{\text{sp}}\{\pi_2(PQ)x_2|Q\in L(H)\}$$

and $\mathscr{A} = P\mathscr{B}(H)P$. Then a mapping $\pi_P\colon \mathscr{A} \to \mathscr{B}(\mathscr{H}_P)$ defined via the formula $\pi_P(A) = \pi_2(A)|\mathscr{H}_P$ is a representation. Let ω_y be a vector functional on $\mathscr{B}(\mathscr{H}_P)$ corresponding to $y = \pi_2(PQ)x_2$. Then $\omega_y \circ \pi_P$ is a normal functional. Indeed, we have

$$\begin{aligned}\omega_y(\pi_P(A)) &= \langle \pi_P(A)\pi_2(PQ)x_2, \pi_2(PQ)x_2\rangle \\ &= \langle \pi_2(APQ)x_2, \pi_2(PQ)x_2\rangle \\ &= \langle \pi_2(QAQ)x_2, x_2\rangle \\ &= f_2(QAQ) \qquad \text{for every } A \in \mathscr{A}\end{aligned}$$

Every increasing net $QF_\alpha Q \nearrow QFQ$, where $(F_\alpha)_\alpha$ is a net of the partial sums of the series $F = \sum_{\alpha\in I} P_\alpha$, F, $F_\alpha \in L(H) \bigcap \mathscr{A}$, can be embedded into some W^*-subalgebra of the form $E\mathscr{B}(H)E$, with $E \in L(H)$ and $\dim E(H) < \nu$. Since f_2 is a normal functional on every subalgebra of this type, we see that the functional $\omega_y \circ \pi_P$ is normal. Because vectors $\pi_2(PQ)x_2$ $[Q \in L(H)]$ span a dense linear submanifold of $\mathscr{H}_{\pi_2}$, we have that π_2 is continuous with respect to a strong operator topology on the unit ball of $\mathscr{A}$. But this is a contradiction with the fact that $\pi_2|\mathscr{K} = 0$. Therefore we have again that $\pi_2(P_{\gamma_1})\pi_2(P_{\gamma_2}) = \pi_2(P_{\gamma_1}P_{\gamma_2}) = 0$ whenever $\gamma_1 \neq \gamma_2$.

Summing up, we have established that $\pi_2(P_\gamma)_{\gamma\in S}$ is a family of nonzero pairwise orthogonal projections acting on $\mathscr{H}_{\pi_2}$ and so $\dim \mathscr{H} \geq |S|$.

Finally, using the fact that x is cyclic for π, we see that

$$\begin{aligned}\dim K \geq \dim \mathscr{R}(m) &= \dim \overline{\mathrm{sp}}\{\pi(P)x | P \in L(H)\} \\ &= \dim \overline{\mathrm{sp}}\{\pi(A)x | A \in \mathscr{B}(H)\} \\ &= \dim (\mathscr{H}_{\pi_1} \oplus \mathscr{H}_{\pi_2}) \geq |S|\end{aligned}$$

This concludes the proof. ■

Let us now consider the problem of when $L(H)$ admits a nontrivial K-valued orthogonal measure. It should be noted that there is no two-valued, finitely additive measure on $L(H)$ (Alda, 1981) [or, equivalently, there is no nonzero orthogonal K-valued measure on $L(H)$ with $\dim K = 1$]. Another result (Kruszynski, 1988) says that $\dim H > \dim K$ if and only if there is no nonzero completely additive (or equivalently, nonsingular σ-additive) measure $m\colon L(H) \to K$. It seems plausible that this result does not hold for finitely additive measures [the finite additivity property is unlikely to determine infinite dimensions—see the conjecture in Kruszynski (1988, 1990)]. In this connection the following theorem seems to be somewhat surprising.

Theorem 2.5. For Hilbert spaces H and K, $\dim H > \dim K$ if and only if there is no nonzero orthogonal measure $m: L(H) \to K$.

Proof. Suppose that $m: L(H) \to K$ is a nontrivial orthogonal measure. According to Kruszynski (1988), if m is completely additive, then $\dim H \leq \dim K$. So we may restrict ourselves to the case when m is not completely additive. Then $\kappa = \dim H \geq \omega$, and using the fact that $|\kappa \times \kappa| = \kappa$, we can find a disjoint family $S \subset [\kappa]^\kappa$ such that $|S| = \kappa$. It is apparent from Lemma 2.4(i) that $\dim K \geq |S| = \dim H$. ■

As a consequence of the foregoing theorem, we obtain a result on the nonexistence of a Jordan homomorphism between some type I W^*-factors, hitherto known for finite-dimensional matrix algebras only (Kruszynski, 1988). [Let us recall that a real linear mapping $\phi: \mathscr{B}(H) \to \mathscr{B}(K)$ is said to be a Jordan homomorphism if $\phi(I) = I$ and $\phi(A^2) = \phi(A)^2$, whenever $A \in S(H)$.]

Corollary 2.6. Let κ and λ be cardinals and let $\mathscr{A}_1$ and $\mathscr{A}_2$ be two W^*-factors of the type I_κ ($\kappa \geq 3$) and I_λ, respectively. Then there exists a Jordan homomorphism $\phi: \mathscr{A}_1 \to \mathscr{A}_2$ if and only if $\kappa \leq \lambda$.

Proof. This result follows easily from Theorem 2.5. Indeed, every Jordan homomorphism $\phi: \mathscr{A}_1 \to \mathscr{A}_2$ induces a nontrivial orthogonal measure $m: L(H) \to K$ ($\dim H = \kappa$, $\dim K = \lambda$) defined by putting

$$m(P) = \phi(P)x, \qquad P \in L(H)$$

where x is an arbitrary nonzero vector in K. ■

Let us now investigate the additivity of orthogonal vector measures. For example (Hamhalter, to appear), if H is separable, then every H-valued orthogonal measure on $L(H)$ is σ-additive (and so it is weakly σ-additive as well as completely additive). We shall show in Proposition 2.9 that this result cannot be fully extended to nonseparable spaces. However, it allows the following generalizations.

Theorem 2.7. Every weakly $\dim H$-additive orthogonal measure $m: L(H) \to H$ is completely additive.

Proof. Looking for a contradiction, let us suppose that m is not completely additive. It is known (Balcar and Štěpánek, 1986) that there is a $\dim H$-almost disjoint family $S \subset \mathscr{P}(\dim H)$ such that $|S| > \dim H$. But then [Lemma 2.5(ii)] we have $\dim H \geq |S| > \dim H$. We have reached a contradiction and thus completed the proof of Theorem 2.7. ■

Theorem 2.8. Suppose that $(\dim H)^{\omega} > \dim H$. Then the following statements hold true:

(i) Every orthogonal measure $m: L(H) \to H$ is weakly σ-additive.

(ii) Every weakly regular orthogonal measure $m: L(H) \to H$ is completely additive.

Proof. (i) Looking for a contradiction, let us suppose that m is not weakly σ-additive. Proposition 1.2 guarantees the existence of an ω-almost disjoint family S on $\dim H$ such that $|S| > \dim H$. Applying Lemma 2.4(ii), we infer that $\dim H \geq |S|$—a contradiction.

(ii) It follows immediately from the statement (i) and Proposition 1.4(ii). ∎

Proposition 2.9. Suppose that $(\dim H)^{\omega} = \dim H$ and $\dim H \geq 2^{\omega}$. Then there is a nonzero free orthogonal measure $m: L(H) \to H$.

Proof. Let $Q \in L(H)$ be a projection onto a separable infinite-dimensional subspace of H. Let g be a state of $\mathscr{B}(Q(H))$ such that g vanishes on all finite-dimensional operators. Let us define a state f of $\mathscr{B}(H)$ by setting $f(A) = g(QAQ|Q(H))$ for every $A \in \mathscr{B}(H)$. Let us consider a representation π of $\mathscr{B}(H)$ on $\mathscr{H}$ obtained by the GNS construction from f. Put $\mathscr{N} = \{A \in \mathscr{B}(H) | f(A^*A) = 0\}$. Then $\mathscr{N}$ is a left null ideal of f. Thus, $\mathscr{H}$ is a completion of the space $\mathscr{B}(H)/\mathscr{N}$ endowed with an inner product $\langle \cdot, \cdot \rangle$ defined by the equality $\langle \hat{A}, \hat{B} \rangle = f(B^*A)$ $[A, B \in \mathscr{B}(H)]$. We prove that $\hat{A} = \hat{Q}A$ for every $A \in \mathscr{B}(H)$. Indeed, $(I-Q) \in \mathscr{N}$ and so $A(I-Q) = A - AQ \in \mathscr{N}$. Making use of the spectral theorem, it can be shown that the set $\{\hat{P} | P \in L(H), \dim P(H) = \omega\}$ generates a dense subspace of $\mathscr{H}$. So $\dim \mathscr{H} \leq |\{\hat{P} | P \in L(H), \dim P(H) = \omega\}|$.

Let us now investigate the size of the set $\mathscr{M}$ of all closed separable infinite-dimensional subspaces of H. Let B be an orthonormal basis of H. Let us denote by M the subset of ${}^{\omega \times \omega}C$ consisting of all matrices $(\alpha_{i,j})_{i,j \in N \times N}$ such that $\sum_{j \in N} |\alpha_{i,j}|^2 < \infty$ for every $i \in N$. Put $\mathscr{N} = M \times {}^{\omega \times \omega}B$. Let us define a mapping $F: \mathscr{N} \to \mathscr{M}$ in the following way: Suppose that $r = [(\alpha_{i,j}), (e_{i,j})] \in \mathscr{N}$. Then we put $F(r) = \overline{\mathrm{sp}}(v_i)_{i \in N}$, where each v_i is of the form $v_i = \sum_{j \in N} \alpha_{i,j} e_{i,j}$. The mapping F is surjective and so

$$|\mathscr{M}| \leq |\mathscr{N}| \leq |{}^{\omega \times \omega}C \times {}^{\omega \times \omega}B| = \max(2^{\omega}, (\dim H)^{\omega}) = \dim H$$

Let $m: L(H) \to \mathscr{H}$ be an orthogonal vector measure given by the formula

$$m(P) = \pi(P)x$$

where $x \in \mathscr{H}$ is a cyclic vector for π. We have $m(P) = 0$, whenever $\dim P(H) < \infty$. Moreover, $\dim \mathscr{R}(m) \leq \dim \mathscr{H} \leq \dim H$ by the preceding estimation. Identifying now $\mathscr{R}(m)$ with a subspace of H, we complete the proof of Proposition 2.9. ∎

Let us remark that the free orthogonal measure constructed in the foregoing proposition is not weakly σ-additive and so the assumption $(\dim H)^{\omega} > \dim H$ in Theorem 2.8 is essential.

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